BULLETIN OF THE BRAZILIAN MATHEMATICAL SOCIETY

Bull Braz Math Soc, New Series 35(3), 453-472

Hopf hypersurfaces in space forms

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Abstract. Some geometrical properties of Hopf hypersurfaces of Kähler manifolds are introduced and a special attention is given to the case of hypersurfaces in complex projective spaces.

Keywords: tubular hypersurfaces, real hypersurfaces, Complex Projectives Spaces.

Mathematical subject classification: 53A20.

1 Introduction

We consider real hypersurfaces of complex projective spaces and also hypersurfaces of more general spaces. A complex structure of the ambient space yields in each of their oriented hypersurfaces a special tangent vector field which is obtained by applying the complex structure of the ambient space to a unit normal vector field defined on the hypersurface. Throughout this article, we name this special vector field as the Hopf vector field of the hypersurface.

A hypersurface of a Kähler manifold is said to be a Hopf hypersurface when the foliation given by its Hopf vector field is geodesic, in other words, when the integral curves of its Hopf vector field are geodesics of the hypersurface.

Some beautiful and elegant studies of these hypersurfaces have already been carried out throughout the last twenty years and to the best of my knowledge it was Yoshiaki Maeda [8] who in 1976 published the first results concerned with these hypersurfaces for the case of the complex projective spaces. In 1982, Cecil and Ryan [4], assuming the constancy of the rank of the focal map of the hypersurface, characterized the Hopf hypersurfaces of the complex projective spaces as open subsets of tubes around complex submanifolds. In 1986, Kimura [7] used the result of Cecil and Ryan and the work of Takagi [10] on homogeneous hypersurfaces of the complex projective spaces to characterize the Hopf hypersurfaces of constant principal curvatures as tubes around some special complex submanifolds of this complex space form. In 1989, Berndt [1] has obtained similar characterizations for the Hopf hypersurfaces of complex hyperbolic space

Received 27 August 2003.

forms. Finally, in 1995 Berndt, Bolton and Woodward [2] gave a complete characterization of the Hopf hypersurfaces of the 6-sphere as tubular hypersurfaces around almost complex curves, these curves being fully classified in [3].

In this article, we investigate Hopf hypersurfaces in more general Riemannian manifolds. We do this in section 2, whereby we start characterizing the complex space forms as the Kähler manifolds all of whose geodesic hyperspheres are Hopf hypersurfaces. We also consider the reflection map and the push maps induced by a hypersurface of a Kähler manifold and then we determine necessary and sufficient conditions to be satisfied by these maps in order that the hypersurface be a Hopf hypersurface.

In section 3, we obtain some geometrical properties of Hopf hypersurfaces of $\mathbb{C}P^n$ which as well as being relevant on their own also strongly suggest that the assumption used by Cecil-Ryan to characterize Hopf hypersurfaces as tubes can actually be proved. This is exactly what is done in section 3, that is, we use all the geometrical understanding achieved about Hopf hypersurfaces of this complex space form in order to prove that if we assume that every continuous principal curvature function defined on the hypersurface admits a corresponding continuous principal vector field then the rank of the focal map of a Hopf hypersurface is indeed constant. We prove this by means of a special construction of vector fields along geodesics normal to the hypersurface. Therefore, our approach to this problem is to deal with the Hopf hypersurface from a quite extrinsic geometrical viewpoint.

Ogur main purpose in this article is to replace the assumption used by Cecil-Ryan in the following theorem by the more natural condition mentioned above.

Theorem 1 ([4]). *Let M be a connected orientable Hopf hypersurface of $\mathbb{C}P^n$ with Hopf principal curvature $\mu = -2\cot(2r)$. Assume that the focal map Φ_r of M has constant rank k on M. Then k is even and each point $q \in M$ has a neighbourhood V such that $\Phi_r(\perp^1 V)$ is a complex submanifold of $\mathbb{C}P^n$ and V lies on the tube of radius r over $\Phi_r(\perp^1 V)$. Furthermore, if M is compact then its focal set $N = \Phi_r(\perp^1 M)$ is a complex submanifold of $\mathbb{C}P^n$ and M lies on the tube of radius r around N. Conversely, every open subset of a tube of constant radius over a complex submanifold of $\mathbb{C}P^n$ is a Hopf hypersurface.*

2 Hopf hypersurfaces of Kähler manifolds

The geodesic hyperspheres of complex space forms are the simplest examples of Hopf hypersurfaces (cf. [1],[2],[4]). Next, we actually show that these examples are typical of complex space forms.

We shall be considering the following terminology throughout this article. Given a Kähler manifold $\overline{M}$ of any dimension greater or equal to two, with metric $\langle , \rangle$ and complex structure J, let M be a real hypersurface of $\overline{M}$, which we shall simply call hypersurface, and let ξ denote a unit normal vector field defined on a neighbourhood $\mathcal{O} \subset M$ of a point $q \in M$. We can use the exponential map of $\overline{M}$ to extend ξ to a local unit vector field $\dot{\gamma}_{(p,\xi)}(s)$ on $\overline{M}$, where $p \in \mathcal{O}$ and $\gamma_{(p,\xi)}(s) = exp_p(s\xi_p)$. The notation $\gamma_{(p,X)}$ shall always mean the geodesic starting at the point p in the direction of X.

Remark 1. It is important at this point, to call the reader's attention to the use of the notations ξ and and γ that we shall be considering throughout this article. If we denote by M^ϵ a tubular neighbourhood of an open subset $\mathcal{O} \subset M$ of a point $q \in M$, diffeomorphic to $\mathcal{O} \times (-\epsilon, \epsilon)$, we can extend the local normal vector field ξ in such a way that it yields the mapping with the same notation $\xi : M^\epsilon \to \mathfrak{X}(M^\epsilon)$, thus the map $\xi = \xi(p, s)$ when restricted to $\mathcal{O} \subset M$ has two variables. In this context when we use the notation $\gamma_{(p,\xi)}(s)$, we need to be careful with the case when we want to consider the point p fixed. In this situation instead of $\gamma = \gamma(p, s)$ being a function of two variables, it is just a function of one variable that we shall simply denote by $\gamma = \gamma(s)$.

Definition 1. *Let M be a hypersurface of a Riemannian manifold $\overline{M}$. Then for some $\epsilon > 0$ and locally on M, we can define the* **reflection map** $\Re$ *on a tubular neighbourhood M^ϵ of an open subset $\mathcal{O}$ of M by putting*

$$\Re(\gamma_{(q,\xi)}(s)) = \gamma_{(q,\xi)}(-s), \tag{1}$$

where $q \in \mathcal{O}$, $s \in (-\epsilon, \epsilon)$ and ξ is a unit normal vector field on $\mathcal{O}$. For each $s \in (-\epsilon, \epsilon)$, we also define the **push map** $\mathfrak{P}_s$ *by*

$$\mathfrak{P}_s(q) = \gamma_{(q,\xi)}(s). \tag{2}$$

We will denote the parallel hypersurfaces of M^ϵ by M_s so that $M^\epsilon = \bigcup_{|s|<\epsilon} M_s$ and the restriction $\Re_s$ of $\Re$ maps M_s into M_{-s}, whilst $\mathfrak{P}_s$ maps M into M_s.

Remark 2. A parallel hypersurface M_s, as above, is also a level hypersurface with respect to the function $\pi_2 \circ \gamma^{-1}$ where π_2 is the projection onto the second variable and γ is the map mentioned in Remark 1.

When M is a hypersurface of a nearly Kähler manifold $\overline{M}$ then on each level hypersurface M_s we can define locally the vector field U_s given by $U_s(\gamma_{(p,\xi)}(s)) = J\dot{\gamma}$.

If we fix a point $p \in M$ then we shall think of this vector field just as a vector field along $\gamma(s)$ and in this case we shall use the notation $U(s)$ instead of U_s. It is easy to see that $U(s)$ is a parallel vector field along $\gamma(s)$. Indeed,

$$\overline{\nabla}_{\dot{\gamma}} U(s) = J(\overline{\nabla}_{\dot{\gamma}} \dot{\gamma}) = 0. \tag{3}$$

Theorem 2. *Let $(\overline{M}, J)$ be a Kahler manifold. Then $\overline{M}$ is a complex space form if and only if every geodesic hypersphere of $\overline{M}$ is a Hopf hypersurface.*

Proof.
($\Longrightarrow$)
It is clear from the results on Hopf hypersurfaces of complex space forms stated in the first chapter that every geodesic hypersphere of these spaces is indeed a Hopf hypersurface.
($\Longleftarrow$)

Given $q \in \overline{M}$ and a unit vector $X \in T_q\overline{M}$, let $\gamma(s) = exp_q(sX)$ be the geodesic of $\overline{M}$ starting at q in the direction X. Then $U(s) = J\dot{\gamma}(s)$ is the Hopf vector at $\gamma(s)$ of the geodesic hypersphere $\mathcal{G}_s$ centred at q and radius s. Thus, if A_s denotes the shape operator of the hypersurface $\mathcal{G}_s$, we have

$$A_s(U(s)) = \alpha_s U(s), \tag{4}$$

where α_s is the Hopf principal curvature of $\mathcal{G}_s$.

Now, we show that the rate of change, in a radial direction, of the shape operators of tubular hypersurfaces satisfies a Riccati differential equation, namely:

$$(\overline{\nabla}_{\dot{\gamma}} A_s)(Z) = A_s^2(Z) + \overline{R}(Z, \dot{\gamma})\dot{\gamma}, \tag{5}$$

where Z is a vector field, orthogonal to $\dot{\gamma}$, defined along γ and $(\overline{\nabla}_{\dot{\gamma}} A_s)(Z) = \overline{\nabla}_{\dot{\gamma}}(A_s Z) - A_s(\overline{\nabla}_{\dot{\gamma}} Z)$. Indeed, equation (5) follows from the definition of the curvature tensor

$$\begin{aligned} \overline{R}(Z, \dot{\gamma})\dot{\gamma} &= \overline{\nabla}_Z \overline{\nabla}_{\dot{\gamma}} \dot{\gamma} - \overline{\nabla}_{\dot{\gamma}} \overline{\nabla}_Z \dot{\gamma} - \overline{\nabla}_{[Z,\dot{\gamma}]} \dot{\gamma} \\ &= \overline{\nabla}_{\dot{\gamma}}(A_s Z) - \overline{\nabla}_{(-A_s Z - \overline{\nabla}_{\dot{\gamma}} Z)} \dot{\gamma} \\ &= \overline{\nabla}_{\dot{\gamma}}(A_s Z) - A_s^2 Z - A_s(\overline{\nabla}_{\dot{\gamma}} Z). \end{aligned}$$

By using (3), (4) and (5), we obtain

$$\overline{R}(U(s), \dot{\gamma})\dot{\gamma} = (\dot{\alpha}_s - \alpha_s^2)U(s). \tag{6}$$

Given a tangential vector $Y \in T_q\overline{M}$ such that Y is orthogonal to both vectors X and JX, let Y_s denote the parallel transport of Y along γ. Then (6) implies $\langle \overline{R}(U(s), \dot{\gamma})\dot{\gamma}, Y_s \rangle = 0$ for any $s \neq 0$ and hence by continuity we have

$$\langle \overline{R}(JX, X)X, Y \rangle = 0. \tag{7}$$

However, it is well known (see for example [9] or [11]) that the condition (7) on the curvature tensor characterizes the complex space forms. □

It is convenient to point out here that the Riccati equation (5) for the second fundamental forms of the tubular hypersurfaces around a submanifold P, encompasses essentially the same information as the Jacobi differential equation which defines Jacobi fields on $\overline{M}$. This equation has been useful to study the geometry of tubular hypersurfaces in general. (c.f. [6] and references mentioned there.)

We remark that in [12], the authors used Jacobi fields to show that the complex space forms are characterized by the fact that their geodesic hyperspheres are quasi-umbilical with respect to their Hopf vector field. Thus the result we have just proved improves this characterization in the sense that, being Hopf hypersurfaces, the geodesic hyperspheres of complex space forms satisfy some further geometrical properties.

Remark 3. The theorem above can be proved also using Jacobi fields instead of the Riccati equation, however, the proof would be less elegant.

Lemma 1. *Let $\sigma(t)$ be a smooth curve of a hypersurface M in $\mathbb{C}P^n$. Let ξ denote a unit normal vector field on M. Then the variational vector field $W(s)$ defined along $\gamma_{(\sigma(t),\xi)}(s)$ by $W(s) = \frac{d}{dt}(\gamma_{(\sigma(t),\xi)}(s))$ satisfies*

$$\mathfrak{R}_*(W(s)) = W(-s) \tag{8}$$

$$\mathfrak{P}_{s*}(\sigma'(t)) = W(s). \tag{9}$$

Proof. Indeed, the lemma follows from direct application of (1) and (2). □

Remark 4. Note that M is the fixed point set of $\mathfrak{R}$ and so if $\mathfrak{R}$ is an isometry then M is a totally geodesic submanifold of $\overline{M}$.

Indeed, this is just a consequence of the well known fact that a connected component of the fixed point set of an isometry of $\overline{M}$ is a totally geodesic submanifold of $\overline{M}$. But we should note that particularly for the reflection map we can also prove this directly. Although the proof we give below is assuming that M is a hypersurface, it can be similarly applied to submanifolds of higher codimension.

We shall make an extensive use of Jacobi fields throughout this work. These have been a powerful tool employed by differential geometers to approach a large range of mathematical issues. It is very easy to find a good wealth of the basic theory about these fields in the literature, however, we shall use here the characterization of a Jacobi field as a variational vector field defined by a geodesic variation and we shall also use the following property.

Lemma 2. *Given $p \in M$, let η denote a local normal vector field defined on M and let A_η be the shape operator of M with respect η. Then a Jacobi vector field $W(s)$ defined along a geodesic $\gamma = \gamma_{(p,\eta)}(s)$ of $\overline{M}$, shall satisfy the conditions*

$$W(0) \in T_pM \qquad \text{and} \qquad \dot{W}(0) + A_\eta(W(0)) \in \perp_p M, \tag{10}$$

if and only if $W(s)$ is the variational vector field corresponding to a geodesic variation $f : (-\epsilon, \epsilon) \times [0, r] \longrightarrow \overline{M}$ of γ complying with the following conditions

$$f(t, 0) \in M \;\; \text{for each } t \in (-\epsilon, \epsilon) \;\; \text{and} \;\; \frac{\partial f}{\partial s}(t, 0) \in \perp_{f(t,0)} M. \tag{11}$$

In this case, we shall say that W is a **<u>M-Jacobi field</u>** *of $\overline{M}$.*

In addition, it is important to highlight here the following facts. Since the M-Jacobi field W is orthogonal to γ, the variation given in the lemma yields locally a surface of $\overline{M}$, which implies $\frac{D}{\partial s}\frac{\partial f}{\partial t} = \frac{D}{\partial t}\frac{\partial f}{\partial s}$ and consequently $[\dot{\gamma}, W] = 0$. Therefore, if ξ denotes a local unit normal field on the tubular hypersurface M_r around M then the corresponding shape operator A_ξ of M_r satisfies:

$$(A_\xi W)(r) = -(\overline{\nabla}_{\dot{\gamma}} W)(r). \tag{12}$$

A proof for Lemma (2) can be found for instance in [5].

Given $q \in M$, let ξ be a local unit normal vector field on M and let $X \in T_qM$ be an eigenvector of A_ξ. Let us consider a curve σ on M with $\sigma(0) = q$ and $\sigma'(0) = X$. Then the geodesic variation $\gamma_{(\sigma,\xi)}$ of the geodesic $\gamma_{(q,\xi)}$ gives the variational vector field $W(s)$ along $\gamma_{(q,\xi)}(s)$ which is a Jacobi field satisfying conditions (10) and so the shape operator A_ξ of M satisfies (12), so that $\dot{W}(0) = -A_\xi(W(0)) = -\lambda W(0)$. Now, if $\Re$ is an isometry then it follows from (8) that the function $|W(s)|^2$ is even. Thus, its derivative is an odd function which implies $\dot{W}(0) = 0$ and hence $\lambda = 0$.

Theorem 3. *If M is a hypersurface of a nearly Kähler manifold $\overline{M}$ satisfying condition $(\star)$ then M is a Hopf hypersurface.*

$(\star)$: *for each $s \in (-\epsilon, \epsilon)$, $\Re$ maps the Hopf vector field of M_s*
to a scalar multiple of the Hopf vector field of M_{-s}.

Proof. Let M be a hypersurface of $\overline{M}$ satisfying the condition in the Theorem. Then in order to prove that M is a Hopf hypersurface, we will just verify that the Hopf vector field U of M is a principal vector field.

Given $q \in M$, consider a local unit normal vector field ξ of M defined around q. Let $A = A_\xi$ denote the second fundamental form of M. It follows from (1) that

$$\mathfrak{R}_*|_{\gamma_{(q,\xi)}(s)} \; (\dot{\gamma}_{(q,\xi)}(s)) = -\dot{\gamma}_{(q,\xi)}(-s). \tag{13}$$

By assumption there exists a smooth function $g(s) = g(q, s)$ such that

$$\mathfrak{R}_*|_{\gamma_{(q,\xi)}(s)} \; (U_s) = g(s)U_{-s}. \tag{14}$$

We can fix the point q because in what follows we shall be considering the rate of change of the function g only in the radial direction. Since $\mathfrak{R}$ is a smooth map, using (3), we have

$$\begin{aligned} &\mathfrak{R}_*[\dot{\gamma}, U_s] = \left[\mathfrak{R}_*\dot{\gamma}, \mathfrak{R}_*U_s\right] \\ \Longrightarrow\; &\mathfrak{R}_*(\overline{\nabla}_{\dot{\gamma}}U_s) - \mathfrak{R}_*(\overline{\nabla}_{U_s}\dot{\gamma}) = \overline{\nabla}_{\mathfrak{R}_*\dot{\gamma}}\mathfrak{R}_*U_s - \overline{\nabla}_{\mathfrak{R}_*U_s}\mathfrak{R}_*\dot{\gamma} \\ \Longrightarrow\; &\overline{\nabla}_{\mathfrak{R}_*U_s}\mathfrak{R}_*\dot{\gamma} = \overline{\nabla}_{\mathfrak{R}_*\dot{\gamma}}\mathfrak{R}_*U_s + \mathfrak{R}_*(\overline{\nabla}_{U_s}\dot{\gamma}). \end{aligned} \tag{15}$$

Now, let A_s denote the shape operator of the level hypersurface M_s with respect to the normal field $\dot{\gamma}(s)$ Then if we substitute (13) and (14) in (15) we obtain

$$g(s)A_{-s}U_{-s} = -\dot{g}(s)U_{-s} - \mathfrak{R}_*(A_sU_s), \tag{16}$$

so that by taking the limit when s goes to zero and recalling that the reflection restricts to the identity map on M, we finally have

$$AU(q) = -\frac{1}{2}\dot{g}(0)U(q). \tag{17}$$

And hence M is a Hopf hypersurface. □

Theorem 4. *If a hypersurface M of a nearly Kahler manifold $\overline{M}$ satisfies the condition $(\star\star)$ then each level hypersurface M_s is a Hopf hypersurface.*

$(\star\star)$: *for each $s \in (-\epsilon, \epsilon)$, $\mathfrak{P}_s$ maps the Hopf vector field of*
M to a scalar multiple of the Hopf vector field of M_s.

Proof. Let us use the same notation and terminology as in the proof of Theorem (2). Here we can give a simpler proof since by using the assumption we see that the push map $\mathfrak{P}_s$ will map the integral curve σ of U to the integral curve, possibly reparametrised, σ_s of U_s. This fact implies that there exists a smooth function $f(s)$ such that the Jacobi field $V(s)$ along $\gamma_{(q,\xi)}(s)$ defined by $V(s) = \frac{d}{dt}\gamma_{(\sigma,\xi)}(s)$ can be expressed by:

$$V(s) = f(s)U_s. \tag{18}$$

Using (3), we obtain $\dot{V} = \dot{f}U_s$. Now, observing that V satisfies the conditions (10), that is, V is a M_s-Jacobi field, we have as a consequence of (12) that

$$f(s)A_sU_s = A_sV(s) = -\dot{V} = -\dot{f}U_s, \tag{19}$$

and hence M_s is a Hopf hypersurface. □

3 Properties of Hopf hypersurfaces in $\mathbb{C}P^n$

We give here some further geometrical properties of Hopf hypersurfaces in $\mathbb{C}P^n$ which not only point out more evidence that they are indeed tubular but also highlight some special features of the geometry of such hypersurfaces.

Let M be a Hopf hypersurface of $\mathbb{C}P^n$. Let ξ and $U = J\xi$ denote a unit local normal field and the corresponding Hopf vector field respectively. The Hopf principal curvature α of M is locally constant ([8]), thus we may consider $\alpha = -2\cot(2r)$, for some constant $r \in (0, \frac{\pi}{4}]$. Moreover, using Gauss and Codazzi equations, Maeda ([8]) has shown the following main result known about the geometry of a Hopf hypersurface of $\mathbb{C}P^n$.

Theorem 5. *Let M be a Hopf hypersurface of $\mathbb{C}P^n$ with Hopf principal curvature α. Let A denote the shape operator of M with respect to a unit normal vector field ξ on M and let ϕ be the tensor on M induced by the complex structure J of $\mathbb{C}P^n$ defined by $\phi(X) = JX - \langle JX, \xi\rangle\xi$. Then these tensors satisfy the following relation*

$$A\phi A = \phi + \frac{\alpha}{2}(A\phi + \phi A). \tag{20}$$

Remark 5. As Maeda observed, the result above shows, in particular, that if X is a principal vector field of M with principal curvature λ and X is orthogonal to the Hopf vector field U then the vector field ϕX is also principal with corresponding principal curvature $\tilde{\lambda}$, where

$$\tilde{\lambda} = \frac{\alpha\lambda + 2}{2\lambda - \alpha}. \tag{21}$$

If we consider the principal curvature λ given in terms of a new function $\theta : M \to \mathbb{R}$ by $\lambda = -\cot(r+\theta)$, where the function θ is chosen so that $-r < \theta < \pi - r$, then it follows from (21) that $\tilde{\lambda} = -\cot(r-\theta)$.

The local constancy of the Hopf principal curvature α of M is not an isolated fact in the sense that by using the Codazzi equation for M we can actually prove the following

Proposition 1. *Let X be a unit smooth principal vector field of a Hopf hypersurface $M \subset \mathbb{C}P^n$ corresponding to a principal curvature function λ. Then λ is constant along any integral curve of the Hopf vector field U.*

Proof. Using that the Hopf principal curvature α is constant, we have from the Codazzi equation

$$U(\lambda)X = \alpha\nabla_X U - A(\nabla_X U) - \lambda\nabla_U X + A(\nabla_U X) - \phi X.$$

Thus, using that $\nabla_X U = -\phi AX$ and (21) we have

$$U(\lambda)X = A(\nabla_U X) - \lambda\nabla_U X - (1 + \alpha\lambda - \lambda\tilde{\lambda})\phi X. \tag{22}$$

Consequently, the inner product of this equation with X yields $U(\lambda) = 0$. □

Remark 6. Let X be a unit smooth principal vector field with corresponding principal curvature λ. If $\alpha = 0$, equation (22) shows that $\nabla_U X$ is a principal vector field corresponding to the same principal curvature λ and if $\alpha \neq 0$, this fact is true if and only if $\lambda = \tilde{\lambda}$ and hence, using (21), if and only if $\lambda = -\cot(r)$ or $\lambda = \tan(r)$.

We shall see now that, in general, almost every principal vector of a level hypersurface of a Hopf hypersurface M in $\mathbb{C}P^n$ is obtained simply by doing the parallel transport of the principal vectors of M along normal geodesics.

Theorem 6. *Let M be a Hopf hypersurface of $\mathbb{C}P^n$. Let X be a unit principal vector at $q \in M$ corresponding to an eigenvalue λ such that X is either equal or orthogonal to the Hopf eigenvector U at q. Then for each s, the parallel transport $X(s)$ of X along the normal geodesic $\gamma = \gamma_{(q,\xi)}(s)$ (that is, $\xi = \dot{\gamma}(0) \in \perp_q M$) starting at q, is a principal vector of the level hypersurface M_s.*

Proof. Let us consider a curve σ of M such that $\sigma(0) = q$ and $\sigma^{'}(0) = X$. Then the 2-parameter family of curves in $\mathbb{C}P^n$ $F(s,t) = \gamma_{(\sigma,\xi)}(s)$ yields the Jacobi field $W(s) = \frac{\partial F}{\partial t}(s,0)$, defined along the geodesic $\gamma(s)$. Then W satisfies the initial conditions $W(0) = X$ and $\dot{W}(0) = -A_0 X = -\lambda X$.

On the other hand, let $f : \mathbb{R} \to \mathbb{R}$ denote either the solution of the differential equation $\ddot{f} = -f$ if X is orthogonal to U, or the solution of $\ddot{f} = -4f$ if $X = U$, satisfying the initial conditions $f(0) = 1$ and $\dot{f}(0) = -\lambda$. Then using the curvature tensor $\overline{R}$ for $\mathbb{C}P^n$ we obtain $\overline{R}(fX(s), \dot{\gamma})\dot{\gamma} = -\ddot{f}X(s)$. Therefore, $fX(s)$ is also a Jacobi field along γ having the same initial conditions as W and hence $W = fX(s)$. However, by construction, W is a M_s-Jacobi field for each s and so $A_s(W(s)) = -\dot{W}(s)$, which implies $A_s X(s) = -\frac{\dot{f}}{f}X(s)$. If we denote the Hopf principal curvature by $\alpha = -2\cot(2r)$ and if we write the principal curvature as $\lambda = -\cot(r+\theta)$ then for the case when X is orthogonal to U, the function f is given by

$$f(s) = \frac{\sin(r+\theta+s)}{\sin(r+\theta)}. \tag{23}$$

□

Note that by applying this result to the Hopf vector field, we have also proved the following

Corollary 1. *The level hypersurfaces of a Hopf hypersurface in $\mathbb{C}P^n$ are also Hopf hypersurfaces.*

Theorem 7. *A hypersurface of $\mathbb{C}P^n$ satisfies either the condition $(\star)$ or $(\star\star)$ if and only if it is a Hopf hypersurface.*

Proof. One direction has already been proved in accordance with Theorems (3) and (4). So if M is a Hopf hypersurface, let us show firstly that the push maps $\mathfrak{P}_s$ satisfy $(\star\star)$. Indeed, given $q \in M$, let σ be the integral curve of the Hopf vector field U which passes through q, say $\sigma(0) = q$. Then it follows from (9) that

$$\begin{aligned} \mathfrak{P}_{s*}\mid_q (U_q) &= \frac{d}{dt}\mid_{t=0} \mathfrak{P}_s(\sigma(t)) \\ &= \frac{\partial F}{\partial t}(s,0) \\ &= f(s)U_s, \end{aligned} \tag{24}$$

From which $(\star\star)$ follows.

Now, for the reflection map we set up $W(s) = fX(s)$ in (8), which yields

$$\Re_*(U_s) = \frac{f(-s)}{f(s)} U_{-s}. \tag{25}$$

Therefore, M also satisfies $(\star)$. $\square$

4 Tubular hypersurfaces in $\mathbb{C}P^n$

We recall here Theorem (1), where Cecil-Ryan, under the assumption of constancy of the rank of the focal map of a hypersurface, characterize the Hopf hypersurfaces of $\mathbb{C}P^n$ as open subsets of tubes around complex submanifolds. The calculation for the derivative of the focal map done by Cecil-Ryan (see [4] Proposition 2.5 for details) shows that this assumption is equivalent to the constancy of the multiplicity of the principal curvature $-\cot(r)$, whenever this value is a principal curvature for the hypersurface. Here we shall denote by G the focal map of a hypersurface M of $\mathbb{C}P^n$ whose notation in Cecil-Rayn article is Φ_r. They have proved the following

Lemma 3. *Let M be a Hopf hypersurface of $\mathbb{C}P^n$. Let $U = J\xi$ be the Hopf vector field of M and let $\alpha = -2\cot(2r)$ be the Hopf principal curvature of M. Then given $q \in M$, we have*

(i) $G_* |_{(q,r\xi)} (U) = 0$

(ii) $G_* |_{(q,r\xi)} (X) = 0$ *whenever $X \in T_q M$ is a principal vector of (M, ξ) corresponding to the principal value $-\cot(r)$.*

(iii) $G_* |_{(q,r\xi)} (X) \neq 0$ *otherwise.*

It is worthwhile highlighting here that from the intrinsic point of view the best geometrical property that we know about a Hopf hypersurface in $\mathbb{C}P^n$ is the elegant result of Maeda stated in Theorem (5). However, this result does not seem to be sufficient to evaluate the behaviour of $-\cot(r)$ as an eigenvalue of the shape operator of the hypersurface.

Now, to determine the integral curve σ of U through a given point $q \in M$, we first note from the lemma above that the focal map of M is constant along the integral curves of the Hopf vector field, that is, $G(\sigma, \xi) \equiv p$.

Next, we consider a geodesic $\gamma = \gamma_{(p,\eta)}$ of $\mathbb{C}P^n$ normal to M at q and connecting the points q and p, where η denotes the tangent vector to γ at the point p. We shall assume γ to be parametrized by the arclength s from p to q and so $\gamma(0) = p$ and $\gamma(r) = q$.

We shall use the Hopf fibration $\pi : S^{2n+1} \to \mathbb{C}P^n$ to lift geometrical objects from $\mathbb{C}P^n$ to the sphere. Let $\tilde{\gamma} = \tilde{\gamma}_{(\tilde{p},\tilde{\eta})}(s)$ denote the horizontal geodesic of the sphere obtained as the lift of γ and let $\tilde{\sigma}$ be the curve in S^{2n+1} obtained as the end points of the geodesics $\widetilde{\gamma}_{(\tilde{p},\tilde{\delta})}$ where

$$\tilde{\delta} = \tilde{\delta}(t) = \cos(\bar{t})\tilde{\eta} + i\,\sin(\bar{t})\tilde{\eta} \qquad \text{and} \qquad \bar{t} = \frac{t}{\sin(r)\cos(r)}\ ,$$

in other words, $\tilde{\sigma}(t) = \widetilde{\gamma}_{(\tilde{p},\tilde{\delta}(t))}(r)$. Let us define the vector $\delta = \delta(t) = \pi_*(\tilde{\delta}(t))$ and the curve $\sigma(t) = \pi(\widetilde{\sigma}(t))$. A simple calculation gives the following

Lemma 4. *The curve $\sigma(t)$ is the integral curve through the point q of the Hopf vector field of the Hopf hypersurface M of $\mathbb{C}P^n$.*

Our main result shall need the special constructions of tangent vector fields X_t and V_t that we now start to describe.

In Corollary (1), we showed that the level hypersurfaces M_s of M are also Hopf hypersurfaces. Thus, we just need to replace r by s in the description above in order to describe the integral curve $\sigma_s(t)$ of the Hopf vector field U_s of the level hypersurface M_s starting at the point $\gamma(s)$. For a fixed value t, we shall use the notation $\gamma_t = \gamma_{(t,\delta)}$

Definition 2. *Given a vector $X_0 \in T_qM$ orthogonal to the Hopf vector $U(q)$, let $X_0(s)$ denote the parallel transport of X_0 from $\gamma(r)$ to $\gamma(s)$ along γ. Then we can construct a smooth vector field X_t along $\sigma(t)$ in two different manners which we shall name hereafter as*

Case I. *The vector field $X_t(r)$ is defined as the parallel transport of $X_0(0)$ along $\gamma_t(s)$ from the point p to the point $\sigma(t)$.*

Case II. *The vector field $X_t(r)$ is defined as the parallel transport along $\gamma_t(s)$ from the point p to the point $\sigma(t)$ of the vector*

$$X_t(0) = \cos\left(\frac{\bar{t}}{2}\right)X_0(0) + \sin\left(\frac{\bar{t}}{2}\right)JX_0(0). \tag{26}$$

Remark 7. The vectors $\{X_0(0), JX_0(0)\}$ are orthogonal to the vectors $\{\eta, J\eta\}$ because $X_0(r)$ and $JX_0(r)$ are both also orthogonal to the vectors $\{\dot{\gamma}_0(r), U_0(q)\}$. Therefore, in both constructions above $X_t(0)$ is orthogonal to δ_t for every t. Thus, by elementary properties of parallel vector fields we have that $X_t(s)$ is orthogonal to $\dot{\gamma}_t(s)$ for each value of t and s. In particular, this makes it clear that $X_t(s)$ is indeed a tangent vector field defined along $\sigma_s(t)$ on the level hypersurface M_s.

Definition 3. *Let us denote the induced Riemannian connection of each hypersurface M_s by the same symbol ∇. Then for each construction of $X_t(s)$ as given in Definition (2), we associate the following vector field*

$$V_t(s) = \nabla_{U_s} X_t(s) + \frac{\alpha_s}{2}\phi X_t(s). \tag{27}$$

The constructions of X_t and V_t may appear artificial at first. However, as we shall see in Theorem (8), they arise quite naturally when considering the case of tubular hypersurfaces.

In the sequel, we shall need to recall some basic facts about Jacobi fields of $\mathbb{C}P^n$ in order to prove our next proposition.

We can use the Hopf fibration $\pi : S^{2n+1} \to \mathbb{C}P^n$ to write down Jacobi fields along a geodesic γ of $\mathbb{C}P^n$ in terms of Jacobi fields along the horizontal geodesic $\widetilde{\gamma}$ of the sphere.

Lemma 5. *The Jacobi field $W(s)$ along $\gamma(s)$ satisfying the initial conditions $W(0) = X$ and $\dot{W}(0) = Y$ is determined by*

$$W(s) = \cos(s)B_X(s) + \sin(s)B_Y(s), \tag{28}$$

where $B_Z(s)$ denotes the image under π_* of the parallel transport $\tilde{B}_{\tilde{Z}}(s)$ of $\tilde{Z}$ along $\tilde{\gamma}(s)$.

Let us first have a close look at tubes around complex submanifolds of $\mathbb{C}P^n$. We shall do this in order to get a good picture of the geometrical relation between the principal curvatures of a tubular hypersurface around a complex submanifold and the principal curvatures of this core.

Proposition 2. *Let M be an open subset of a tube $\Phi_r(\perp^1 N)$ of radius r around a complex submanifold N^m of $\mathbb{C}P^n$. The principal vectors of M at a point $q = \gamma(r) = \gamma_{(p,\eta)}(r)$ are obtained according to the following cases*

- **(i)** $A_\xi B = -2\cot(2r)B$, *where $B(s) = \pi_*(\tilde{B}_{i\tilde{\eta}})$ and $W(s) = \sin(s)B(s)$ is the Jacobi field along γ satisfying the initial conditions $W(0) = 0$ and $\dot{W}(0) = J\eta$. Note that $B = J\xi = U$.*

- **(ii)** $A_\xi B = -\cot(r)B$, *where $B(s) = \pi_*(\tilde{B}_{\tilde{X}})$ and $W(s) = \sin(s)B(s)$ is the Jacobi field along γ satisfying the initial conditions $W(0) = 0$ and $\dot{W}(0) = X \in (\perp_p N) \cap \{\eta\}^\perp$*

(iii) $A_\xi B = -\cot(r+\theta)B$,
where X is a principal vector of the shape operator A_η of N corresponding to the principal value $\cot(\theta)$, $B(s) = \pi_*(\tilde{B}_{\tilde{X}})$ *and* $W(s) = \big(\cos(s) - \cot(\theta)\sin(s)\big)B(s)$ *is the Jacobi field along γ satisfying the initial conditions* $W(0) = X \in T_pN$ *and* $\dot{W}(0) = -A_\eta X = -\cot(\theta)X$.

Proof. The proposition follows immediately from Lemma (5) and the fact that the Jacobi field W satisfies $\dot{W}(0) = -A_\xi W(0)$. □

Remark 8. In the Proposition above we can highlight some useful facts. The first, being that **(i)** shows that every tube around a complex submanifold is indeed a Hopf hypersurface. Secondly, it follows from **(ii)** that the multiplicity of the eigenvalue $-\cot(r)$ is exactly $2(n-m)$ at each point of the hypersurface M.

The next theorem points out the geometrical relevance of the vector field V_t for the study of the principal curvatures in the case of a tubular hypersurface.

Theorem 8. *Let M be an open subset of the tube $\Phi_r(\perp^1 N)$ of radius r around a complex submanifold N of $\mathbb{C}P^n$. Let $q = \gamma_{(p,\eta)}(r) \in M$ and let $X_0 \in T_qM$ be a vector orthogonal to $U(q)$. Then the vector fields X_t and V_t, as given in Definitions (2) and (3) satisfy the following properties*

(i) *If X_0 is an eigenvector of M corresponding to the eigenvalue* $-\cot(r)$ *(respectively,* $-\cot(r+\theta)$*). Then for every $s \in (0,r]$, the vector field $X_t(s)$ constructed in case I (case II) is a principal field along σ_s corresponding to the eigenvalue* $-\cot(s)$ *(respectively,* $-\cot(s+\theta)$*).*

(ii) $V_t(s) \neq 0$ *for every $s \in (0,r]$, in Case I.*

(iii) $V_t(s) \equiv 0$, *in case II and consequently* $\nabla_{U_s} X_t = -\frac{\alpha_s}{2}\phi X_t$.

Proof. Item (i), for Case I, follows imediately from item (ii) of Proposition (2). To prove (i) for Case II, we note that from item (iii) of Proposition (2), we have $\overline{\nabla}_{X_0(0)}\eta = -\cot(\theta)X_0(0)$, which implies

$$\overline{\nabla}_{X_t(0)}\delta_t = -\cot(\theta)X_t(0). \tag{29}$$

And hence, using again item (iii) of that proposition, (i) follows.

Now, in order to prove (ii) and (iii), we need to consider the geodesic variation $F(s,t) = \gamma_{(p,\delta_t)}(s)$ with its corresponding variational Jacobi field $W_t(s) =$

$\frac{\partial F}{\partial t}(s,t)$. Then for each t, W_t is a M_s-Jacobi field and hence $A_s W_t = -\dot{W}_t$, where A_s denotes the shape operator of the level hypersurface M_s.

Using Proposition (4), we see that W_t satisfies the initial conditions

$$W_t(r) = U(\sigma(t)) \quad \text{and} \quad \dot{W}_t(r) = 2\cot(2r)U(\sigma(t)).$$

Therefore, setting $h(s) = \frac{\sin(2s)}{\sin(2r)}$, we have $W_t(s) = h(s)U_s$ because $h(s)U_s$ is also a Jacobi field along γ_t satisfying the same initial conditions.

Using the definition of the curvature tensor $\overline{R}$ of $\mathbb{C}P^n$ we have,

$$\overline{R}(\dot{\gamma}_t, W_t)X_t = -2h\phi X_t, \tag{30}$$

which together with the following fact

$$[\dot{\gamma}_t, W_t] = \overline{\nabla}_{\dot{\gamma}_t}(hU_s) - \overline{\nabla}_{hU_s}(\dot{\gamma}_t) = (\alpha_s h + \dot{h})U = 0, \tag{31}$$

yields

$$\overline{\nabla}_{\dot{\gamma}_t}\overline{\nabla}_{W_t}X_t = -2h\phi X_t = \overline{\nabla}_{\dot{\gamma}_t}\left(\frac{\cos(2s)}{\sin(2r)}\phi X_t\right). \tag{32}$$

Therefore we have

$$\overline{\nabla}_{\dot{\gamma}_t}(hV_t) = \overline{\nabla}_{\dot{\gamma}_t}\left(\overline{\nabla}_W X_t + \frac{\alpha_s h}{2}\phi X_t\right) = 0.$$

In other words, the vector field $h(s)V_t(s)$ is parallel along the geodesic $\gamma_t(s)$. Therefore, using this parallelism, (ii) follows from the limit

$$\lim_{s\to 0}(hV_t) = -\frac{1}{\sin(2r)}JX_t \neq 0,$$

and (iii) follows from (26) and the limit $\lim_{s\to 0}(hV_t) = \dfrac{dX_t}{dt} - \dfrac{1}{\sin(2r)}JX_t = 0.$ □

Remark 9. The parallelism of the vector field hV_t is equivalent to the property $\overline{\nabla}_{\dot{\gamma}_t}V_t = \alpha_s V_t$ because of $h(s)$ satisfying $\dot{h}(s) = -\alpha_s h(s)$.

In each construction given by Definition (2), the vector field $X_t(s)$ satisfies the following basic property.

Lemma 6. *The vector field $X_t(s)$ is orthogonal to the Hopf vector field U_s.*

Proof. Let $W_t(s)$ be the M_s-Jacobi field defined along γ_t as in Theorem (8). Then using that $W_t(s) = h(s)U_s$ we have

$$h(s)\langle X_t, U_s\rangle = \langle X_t, W_t\rangle,$$

which by differentiation with respect to s yields

$$\dot{h}(s)\langle X_t, U_s\rangle = \langle X_t, \overline{\nabla}_{\dot{\gamma}} W_t\rangle = \langle X_t, \overline{\nabla}_{W_t}\dot{\gamma}\rangle \tag{33}$$

Thus, calculating the limit of (33) when s goes to zero we obtain

$$\lim_{s\to 0}\langle X_t, U_s\rangle = \lim_{s\to 0}\frac{1}{\dot{h}(s)}\langle X_t, \overline{\nabla}_{W_t}\dot{\gamma}\rangle = \frac{\sin(2r)}{2}\left\langle X_0(0)\frac{d\delta_t}{dt}\right\rangle = 0.$$

Therefore, using again that X_t and U_s are parallel along γ_t, we have $\langle X_t, U_s\rangle \equiv 0$ which proves the Lemma. □

Inspired by the geometrical properties of the tubular hypersurfaces of $\mathbb{C}P^n$ described above, we can show now that these properties hold in general for any Hopf hypersurface of this space form. Thus, we shall be henceforth considering M as an arbitrary Hopf hypersurface of $\mathbb{C}P^n$.

We shall prove next that in the case of a Hopf hypersurface M, the vector fields X_t and V_t also satisfy properties similar to those obtained in Theorem (8) for tubular Hopf hypersurfaces.

Theorem 9. *Let M be a Hopf hypersurface of $\mathbb{C}P^n$. Let $q = \gamma_{(p,\eta)}(r) \in M$ and let $X_0 \in T_qM$ be a vector orthogonal to $U(q)$. Then the vector fields X_t and V_t, as given in Definitions (2) and (3) satisfy the following properties*

- **(i)** $V_t(s) \neq 0$ *for every* $s \in (0, r]$, *in Case I.*
- **(ii)** $V_t(s) \equiv 0$, *in case II and consequently* $\nabla_{U_s} X_t = -\frac{\alpha_s}{2}\phi X_t$.
- **(iii)** *If in addition we assume that M is analytic and X_0 is an eigenvector of M corresponding to the eigenvalue $-\cot(r)$ (respectively, $-\cot(r+\theta)$) then for every $s \in (0, r]$, the vector field $X_t(s)$ constructed in case I (case II) is a principal field along σ_s corresponding to the eigenvalue $-\cot(s)$ (respectively, $-\cot(s+\theta)$).*

Proof. The proof of item (i) and (ii) can be carried out in the same manner as that given in Theorem (8) for tubular hypersurfaces as far as we can show that the vector field $h(s)V_t$ here is also parallel. According to the Remark (9), this parallelism is equivalent to $\overline{\nabla}_{\dot{\gamma}_t} V_t = \alpha_s V_t$, which can be proved as follows.

The vector field X_t is orthogonal to $\dot{\gamma}_t$ by Remark (7) and is also orthogonal to U_s by Lemma (6). Thus, we have

$$\overline{\nabla}_{U_s} X_t = \nabla_{U_s} X_t. \tag{34}$$

We have proved in Theorem (1) that M_s is a Hopf hypersurface and so

$$[\,\dot{\gamma}_t, U_s\,] = \overline{\nabla}_{\dot{\gamma}_t} U_s - \overline{\nabla}_{U_s} \dot{\gamma}_t = \alpha_s U_s. \tag{35}$$

Thus using the results above we obtain

$$\begin{aligned} \overline{\nabla}_{\dot{\gamma}_t} \overline{\nabla}_{U_s} X_t &= \overline{\nabla}_{U_s} \overline{\nabla}_{\dot{\gamma}_t} X_t + \overline{\nabla}_{[\,\dot{\gamma}_t, U_s\,]} X_t + \overline{R}(\dot{\gamma}_t, U_s) X_t \\ &= -2\phi X_t + \alpha_s \nabla_{U_s} X_t. \end{aligned} \tag{36}$$

Now, applying (34),(35) and (36) to (27), we have

$$\overline{\nabla}_{\dot{\gamma}_t} V_t = \left(\frac{\dot{\alpha}_s}{2} - 2 \right) \phi X_t + \alpha_s \nabla_{U_s} X_t = \alpha_s V_t, \tag{37}$$

where for the last equality we have used $\dot{\alpha}_s = 4 + \alpha_s^2$. The last part of the theorem shall be proved by showing that the following vector field $Z_t(s)$ defined along $\sigma_s(t)$ is identically zero.

$$Z_t = A_s X_t - \lambda X_t, \tag{38}$$

where A_s denotes the shape operator of the level hypersurface M_s.

First, we notice that the analyticity of the ambient space $\mathbb{C}P^n$ and of the hypersurface M imply that we can construct a local analytic unit normal field on M. Thus, the field Z_t is also analytic and it is identically zero if and only if all the derivatives of Z_t with respect to t vanish at $t = 0$.

In order to simplify our notation we shall omit any subscript s since it is clear that we are considering all the geometrical objects involved as defined on each level hypersurface M_s.

It follows from the Codazzi equation and the fact that M_s is a Hopf hypersurface that

$$\nabla_U (AX_t) = \nabla_{X_t}(AU) - A(\nabla_{X_t} U) + A(\nabla_U X_t) - \phi X_t. \tag{39}$$

Using the following property of the tensor ϕ

$$(\nabla_X \phi) Y = \langle AX, Y \rangle U - \langle Y, U \rangle AX, \tag{40}$$

we can also differentiate $\phi A X_t$ obtaining

$$\nabla_U (\phi A X_t) = \phi \nabla_U (A X_t). \tag{41}$$

We shall first consider the situation when $AX_0 = -\cot(r+\theta)X_0$ and X_t is constructed as in Case II. Then recalling that the Hopf principal curvature α is constant, we have from (39), item (ii) and (20) that

$$\begin{aligned}\nabla_U(AX_t) &= -\alpha\phi AX_t + A\phi AX_t - \frac{\alpha}{2}A\phi X_t - \phi X_t \\ &= -\frac{\alpha}{2}\phi AX_t\end{aligned} \tag{42}$$

and hence (41) can be simplified to

$$\nabla_U(\phi AX_t) = \frac{\alpha}{2}AX_t. \tag{43}$$

Thus, it follows from (42) and (43) that the n-th derivative of AX_t is given by

$$\nabla_U^n(AX_t) = \begin{cases}(-1)^{m+1}(\frac{\alpha}{2})^n\phi AX_t & \text{if } n = 2m+1. \\ (-1)^m(\frac{\alpha}{2})^n AX_t & \text{if } n = 2m.\end{cases} \tag{44}$$

On the other hand, it follows from item (ii) and (40) that the n-th derivative of X_t is given by

$$\nabla_U^n(X_t) = \begin{cases}(-1)^{m+1}(\frac{\alpha}{2})^n\phi X_t & \text{if } n = 2m+1. \\ (-1)^m(\frac{\alpha}{2})^n X_t & \text{if } n = 2m.\end{cases} \tag{45}$$

Therefore, it follows from (44), (45) and the assumption $AX_0 = \lambda X_0$, that all the derivatives of Z_t at $t = 0$ vanish.

The proof for the situation when $AX_0 = -\cot(r)X_0$ and X_t is constructed as in Case I, is now just a consequence of the previous case. Indeed, in accordance with Theorem (5), if Y_0 is any eigenvector at q corresponding to a principal curvature $\cot(r+\theta)$ with $\theta \neq 0$ then X_0 is orthogonal to both vectors Y_0 and JY_0 since the eigenvalues are all distinct. Consequently, the parallel transport X_t along γ_t remains orthogonal to the parallel transport Y_t of the rotated vector

$$Y_t(0) = \cos\left(\frac{\bar{t}}{2}\right)Y_0(0) + \sin\left(\frac{\bar{t}}{2}\right)JY_0(0).$$

Thus, the vector $X_t(s)$ must lie in the eigenspace $V_{-\cot(r)}$. □

Theorem 10. *Let M be a connected Hopf hypersurface of $\mathbb{C}P^n$ with Hopf principal curvature being $-2\cot(2r)$. Let X be a continuous principal vector field on M corresponding to a continuous principal curvature function $\lambda : M \to \mathbb{R}$. If λ assumes the value $-\cot(r)$ at a particular point $q_0 \in M$ then λ is constant.*

Proof. The set of points where the function λ assumes the value $-\cot(r)$ is certainly closed and so because of M being connected we can obtain our theorem by proving that this set is also open. Next, we shall prove this by contradiction.

Let us assume the existence of a sequence of points $q_n \in M$ converging to $q_0 \in M$ such that at each of these points we have $\lambda_n = \lambda(q_n) \neq -\cot(r)$. Thus, if we define λ in terms of a new function θ by putting $\lambda = -\cot(r+\theta)$, our assumption can be read as $\lim \theta_n = 0$.

For each $n \in \{0, 1, \ldots\}$ we shall denote the integral curve of the Hopf vector field starting at the point q_n by $\sigma^n(t)$. Along each of these curves we can apply the construction given in Definition (2) to obtain a vector field X_t^n satisfying the initial condition $X_t^n(r) = X(q_n)$.

Thus, using Definition (3), we have vector fields X_t^n and V_t^n satisfying the properties stated in Theorem (9), that is, for each t we have $V_t^0 \neq 0$ and for each $n \neq 0$ we have $V_t^n \equiv 0$.

Now, since the vector field V_t^n depends continuously on the initial condition given for the vector field X_t^n, we must have

$$\lim_{n\to\infty} V_t^n(q_n) = V_t^0(q),$$

which contradicts the properties satisfied by these vector fields that we have just mentioned. □

Corollary 2. *Let M be a connected Hopf hypersurface of $\mathbb{C}P^n$ such that every continuous principal curvature function on M corresponds to a continuous principal vector field. If $-\cot(r)$ is an eigenvalue at a point of M then it will be an eigenvalue at any point of M with the same multiplicity.*

Proof. If we order the principal curvatures of M at each point as

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{2n-1},$$

then each λ_j is a continuous principal curvature function and using the theorem above we see that if λ_j assumes the value $-\cot(r)$ at some point then it must be constant and hence $-\cot(r)$ must have constant multiplicity. □

Corollary 3. *Let M be a connected Hopf hypersurface of $\mathbb{C}P^n$ such that to every continuous principal curvature function there corresponds a continuous principal vector field. Then M lies in a tube around a complex submanifold of $\mathbb{C}P^n$.*

Proof. The result follows from Corollary (2) and Theorem (1). □

We also note the important fact that the lift of Hopf hypersurfaces under a holomorphic Riemannian submersion $\pi : \widetilde{W} \to W$ are also Hopf hypersurfaces. This can provide us with a means to obtain examples of Hopf hypersurfaces in more general Kähler manifolds which could possibly be non-tubular hypersurfaces.

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